Exact Traveling Wave Solutions for the Modified Kawahara Equation

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Z. Naturforsch. **60a,** 139 – 144 (2005); received October 21, 2004

The mapping method is used with the aid of the symbolic computation system Mathematica for constructing exact solutions of the modified Kawahara equation. By this method the modified Kawahara equation is investigated and new exact traveling wave solutions are obtained. The solutions obtained in this paper include Jacobi elliptic solutions, combined Jacobi elliptic solutions, solitary wave solutions, periodic wave solutions, trigonometric solutions and rational solutions.

Key words: Modified Kawahara Equation; Periodic Wave Solution; Mapping Method; Jacobi Elliptic Functions.

1. Introduction

In the last decades, direct search for exact solutions of nonlinear partial differential equations (NLPDEs) has become increasingly attractive partly due to the availability of computer symbolic software like Mathematica or Maple, which allows to perform complicated and tedious algebraic calculations and helps to find exact solutions of NLPDEs [1 – 4]. There has been much activity aiming at finding powerful methods for obtaining such solutions. We can cite the Backlund transformation [5], the Darboux transformation [6], the Jacobi elliptic function method [7], the tanh-function method [8], the sine-cosine function method [9, 10], the homogenous balance method [11], and the Jacobi function expansion method [12, 13]. Very recently, a unified method called the mapping method has been developed to obtain Jacobi elliptic functions, solitons and periodic solutions to some NLPDEs [14 – 16]. A remarkable observation about the mapping method is that it allows to find both Jacobi elliptic functions, triangular functions and solitons using the same procedure. Moreover, this method permits the classification of solutions depending on four parameters.

The motivation of this paper is to use the mapping method for constructing new traveling wave solutions to the modified Kawahara equation.

2. Elliptic Equation and its Solutions

The main idea of the mapping method is to take full advantage of the elliptic equation that Jacobi elliptic functions satisfy and use its solutions to obtain new periodic and solitonic solutions of the modified Kawahara equation. The basic idea is as follows. For a given nonlinear partial differential equation

$$N(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{1}$$

we seek its traveling wave solution of the form

$$u(x,t) = u(\zeta), \quad \zeta = kx - \lambda t. \tag{2}$$

Substituting (2) into (1) yields an ordinary differential equation of $u(\zeta)$. Then $u(\zeta)$ is expanded into a polynomial in $f(\zeta)$,

$$u(\zeta) = \sum_{i=0}^{n} a_i f^i, \tag{3}$$

where a_i are constants to be determined and n is fixed by balancing the linear term of the highest order derivative with the nonlinear term. In the present work, we shall introduce the following new auxiliary ordinary differential equation

$$f'(\zeta) = \sqrt{p f^2(\zeta) + q f^4(\zeta) + s f^6(\zeta) + r}. \tag{4}$$

Here the prime means derivatives with respect to ζ and p, q, s, and r are constants. After (3) with (4) is substituted into the ordinary differential equation, the coefficients a_i, k, λ, p, q, s, and r may be determined. Thus (3) establishes an algebraic mapping relation between the solution of (1) and that of (4). We shall construct exact solutions for the modified Kawahara equation by using the solutions of (4) shown in Table 1.

Table 1. Different solutions of the elliptic equation (4).

Case	Arbitrary constants	Solutions of (1)
1	$p=-(1+m^2)$, $q=2m^2$, $s=0$, $r=1$	$f(\zeta)=\mathrm{sn}\zeta$, $f(\zeta)=\mathrm{cd}\zeta$
2	$p=2m^2-1$, $q=-2m^2$, $s=0$, $r=1-m^2$	$f(\zeta)=\mathrm{cn}\zeta$
3	$p=2-m^2$, $q=-2$, $s=0$, $r=m^2-1$	$f(\zeta)=\mathrm{dn}\zeta$
4	$p=2m^2-1$, $q=2$, $s=0$, $r=-m^2(1-m^2)$	$f(\zeta)=\mathrm{ds}\zeta$
5	$p=2-m^2-1$, $q=2$, $s=0$, $r=1-m^2$	$f(\zeta)=\mathrm{cs}\zeta$
6	$p=\frac{m^2-2}{2}$, $q=\frac{m^2}{2}$, $s=0$, $r=\frac{1}{4}$	$f(\zeta)=\frac{\mathrm{sn}\zeta}{1\pm\mathrm{dn}\zeta}$
7	$p=\frac{m^2-2}{2}$, $q=\frac{m^2}{2}$, $s=0$, $r=\frac{m^2}{4}$	$f(\zeta)=\mathrm{sn}\zeta\pm\mathrm{icn}\zeta$, $\mathrm{i}^2=-1$,
8	$p=\frac{1-2m^2}{2}$, $q=\frac{1}{2}$, $s=0$, $r=\frac{1}{4}$	$f(\zeta)=\frac{\mathrm{dn}\zeta}{\mathrm{i}\sqrt{1-m^2}\mathrm{sn}\zeta\pm\mathrm{dn}\zeta}$, $f(\zeta)=\frac{\mathrm{dn}\zeta}{m\mathrm{cn}\zeta\pm\mathrm{i}\sqrt{1-m^2}}$, $f(\zeta)=\frac{\mathrm{cn}\zeta}{\sqrt{1-m^2}\mathrm{sn}\zeta\pm\mathrm{dn}\zeta}$, $f(\zeta)=\frac{\mathrm{sn}\zeta}{1\pm\mathrm{cn}\zeta}$, $f(\zeta)=m\mathrm{sn}\zeta\pm\mathrm{idn}\zeta$
9	$p=\frac{m^2+1}{2}$, $q=\frac{m^2-1}{2}$, $s=0$, $r=\frac{m^2-1}{4}$	$f(\zeta)=\frac{\mathrm{dn}\zeta}{1\pm m\mathrm{sn}\zeta}$
10	$p=\frac{1+m^2}{2}$, $q=\frac{1-m^2}{2}$, $s=0$, $r=\frac{1-m^2}{4}$	$f(\zeta)=\frac{\mathrm{cn}\zeta}{1\pm\mathrm{sn}\zeta}$
11	$p=\frac{1+m^2}{2}$, $q=-\frac{1}{2}$, $s=0$, $r=-\frac{(1-m^2)^2}{4}$	$f(\zeta)=m\mathrm{cn}\zeta\pm\mathrm{dn}\zeta$
12	$p=\frac{1+m^2}{2}$, $q=\frac{(1-m^2)^2}{2}$, $s=0$, $r=\frac{1}{4}$	$f(\zeta)=\frac{\mathrm{sn}\zeta}{\mathrm{dn}\zeta\pm\mathrm{cn}\zeta}$
13	$p=\frac{m^2-2}{2}$, $q=\frac{m^2}{2}$, $s=0$, $r=\frac{1}{4}$	$f(\zeta)=\frac{\mathrm{cn}\zeta}{\sqrt{1-m^2}\pm\mathrm{dn}\zeta}$
14	$p=0$, $q=2$, $s=0$, $r=0$	$f(\zeta)=C/\zeta$

Table 2. Jacobi elliptic functions degenerate into hyperbolic functions when the modulus is approaching 1.

$\mathrm{sn}\zeta\to\tanh\zeta$	$\mathrm{cn}\zeta\to\mathrm{sech}\zeta$	$\mathrm{dn}\zeta\to\mathrm{sech}\zeta$	$\mathrm{sc}\zeta\to\sinh\zeta$
$\mathrm{sd}\zeta\to\sinh\zeta$	$\mathrm{cd}\zeta\to 1$	$\mathrm{dc}\zeta\to 1$	$\mathrm{ns}\zeta\to\coth\zeta$
$\mathrm{nd}\zeta\to\cosh\zeta$	$\mathrm{cs}\zeta\to\mathrm{csch}\zeta$	$\mathrm{ds}\zeta\to\mathrm{csch}\zeta$	$\mathrm{nc}\zeta\to\cosh\zeta$

Table 3. Jacobi elliptic functions degenerate into trigonometric functions when the modulus is approaching 0.

$\mathrm{sn}\zeta\to\sin\zeta$	$\mathrm{cn}\zeta\to\cos\zeta$	$\mathrm{dn}\zeta\to 1$	$\mathrm{sc}\zeta\to\tan\zeta$
$\mathrm{sd}\zeta\to\sin\zeta$	$\mathrm{cd}\zeta\to\cos\zeta$	$\mathrm{ns}\zeta\to\csc\zeta$	$\mathrm{nc}\zeta\to\sec\zeta$
$\mathrm{nd}\zeta\to 1$	$\mathrm{cs}\zeta\to\cot\zeta$	$\mathrm{ds}\zeta\to\csc\zeta$	$\mathrm{dc}\zeta\to\sec\zeta$

Here C is a constant. The Jacobi elliptic functions $\mathrm{sn}(\zeta|m)$, $\mathrm{cn}(\zeta|m)$, $\mathrm{dn}(\zeta|m)$, where m $(0<m<1)$ is the modulus of the elliptic function, are doubly periodic and possess properties of triangular functions. In addition we see that other solutions are obtained from Table 1 in case of degeneracy. As we know, when $m\to 1$, Jacobi elliptic functions degenerate as hyperbolic functions as indicated in Table 2. When $m\to 0$, Jacobi elliptic functions degenerate into trigonometric functions as shown in Table 3.

3. The Modified Kawahara Equation

We consider the modified Kawahara equation

$$u_t+u_x+u^2u_x+\alpha u_{xxx}+\beta u_{xxxxx}=0, \tag{5}$$

where α and β are nonzero real constants. This equation arises in the theory of shallow water waves [17], and its exact solutions were obtained by using the tanh-function method [18] and the sech-function method [19]. After using the transformation $u(x,t)=u(\zeta)$, $\zeta=x-\lambda t$, and integrating once, (5) becomes

$$(1-\lambda)u+\frac{1}{3}u^3+\alpha u''+\beta u^{(4)}=0. \tag{6}$$

The solution of (6) may be chosen as

$$u(\zeta)=\sum_{i=0}^{n}a_i f^i(\zeta), \tag{7}$$

with arbitrary constants a_i $(i=0,1,\dots n)$ to be determined later. Balancing the highest derivative term $u^{(4)}$ with the highest power nonlinear term u^3 gives the leading order $n=2$. Substituting (7) into (6) along with (4) and using Mathematica yields a system of equations with respect to $f^i(\zeta)$. Setting the coefficients of $f^i(\zeta)$ in the obtained system of equations to zero, we get the following set of algebraic equations

$$a_0^3+6a_2r(\alpha+4p\beta)-3a_0(\lambda-1)-3c_1=0,$$

$$3a_1(1+a_0^2+p\alpha+p^2\beta+6qr\beta-\lambda)=0,$$
$$a_0a_1^2+a_0^2a_2+a_2(1+4p\alpha+16p^2\beta+36qr\beta-\lambda)=0,$$
$$a_1(a_1^2+6a_0a_2+3q\alpha+30pq\beta+60rs\beta)=0,$$
$$3a_2(a_1^2+a_0a_2+3q\alpha+60pq\beta+80rs\beta)=0,$$
$$3a_1[a_2^2+6q^2\beta+s(\alpha+26p\beta)]=0,$$
$$a_2\{a_2^2+2[45q^2\beta+4s(\alpha+40p\beta)]\}=0,$$
$$60s\beta a_1q=0,$$
$$240s\beta a_2q=0,$$
$$35s^2\beta a_1=0,$$
$$128s^2a_2\beta=0. \tag{8}$$

Solving the above system, we get

1.

$$a_0=\frac{15c_1\beta(\alpha+20p\beta)}{\alpha^3-120(2p^2-3qr)\alpha\beta^2+800p(4p^2-9qr)\beta^3},$$
$$a_1=0,$$
$$a_2=\frac{450c_1q\beta^2}{\alpha^3-120(2p^2-3qr)\alpha\beta^2+800p(4p^2-9qr)\beta^3},$$
$$c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{(\alpha^3-120(2p^2-3qr)\alpha\beta^2+800p(4p^2-9qr)\beta^3)^2}{\beta^3}},$$
$$\lambda=1-\frac{\alpha^2}{10\beta}-24p^2\beta+36qr\beta. \tag{9a}$$

2.

$$a_0=\frac{3(3c_1p^4-2c_1p^2qr)}{(171p^6-294p^4qr-204p^2q^2r^2-40q^3r^3)\beta^3},$$
$$a_1=0,$$
$$a_2=\frac{18c_1p^3q}{(171p^6-294p^4qr-204p^2q^2r^2-40q^3r^3)\beta^3},$$
$$c_1=\frac{1}{3}\sqrt{\frac{5}{2}}\sqrt{-\frac{(-171p^6-294p^4qr-204p^2q^2r^2-40q^3r^3)^2\beta^3}{p^6}},$$
$$2\lambda=2-53p^2\beta-+52qr\beta-\frac{20q^2r^2\beta}{p^2},$$
$$\alpha=-5p-\frac{10qr}{p}. \tag{9b}$$

Substituting (9a) into (7) and using the solution of (4), we obtain:

Case 1

$$u=\frac{15c_1\beta[\alpha+20(-1-m^2)\beta]}{A_1}+\frac{900c_1m^2\beta^2}{A_1}\mathrm{sn}^2(\zeta),$$
$$\lambda=1-\frac{\alpha^2}{10\beta}+72m^2\beta-24(-1-m^2)^2\beta,$$
$$c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_1^2}{\beta^3}},$$
$$A_1=\alpha^3-120[-6m^2+2(-1-m^2)^2]\alpha\beta^2$$
$$+800(-1-m^2)[-18m^2+4(-1-m^2)^2]\beta^3.$$

Case 2

$$u=\frac{15c_1\beta[\alpha+20(-1+2m^2)\beta]}{A_2}-\frac{900c_1m^2\beta^2}{A_2}\mathrm{cn}^2(\zeta),$$
$$\lambda=1-\frac{\alpha^2}{10\beta}-72m^2(1-m^2)\beta-24(-1+2m^2)^2\beta,$$
$$c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_2^2}{\beta^3}},$$
$$A_2=\alpha^3-120[6m^2(1-m^2)+2(-1+2m^2)^2]\alpha\beta^2$$
$$+800(-1+2m^2)[18m^2(1-m^2)+4(-1+2m^2)^2]\beta^3.$$

Case 3

$$u=\frac{15c_1\beta[\alpha+20(2-m^2)\beta]}{A_3}-\frac{900c_1\beta^2}{A_3}\mathrm{dn}^2(\zeta),$$
$$\lambda=1-\frac{\alpha^2}{10\beta}-24(2-m^2)^2\beta-72(-1+m^2)\beta,$$
$$c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_3^2}{\beta^3}},$$
$$A_3=\alpha^3-120[2(2-m^2)^2+6(-1+m^2)]\alpha\beta^2$$
$$+800(2-m^2)[4(2-m^2)^2+18(-1+m^2)]\beta^3.$$

Case 4

$$u=\frac{15c_1\beta[\alpha+20(-1+2m^2)\beta]}{A_4}+\frac{900c_1\beta^2}{A_4}\mathrm{ds}^2(\zeta),$$
$$\lambda=1-\frac{\alpha^2}{10\beta}-72m^2(1-m^2)\beta-24(-1+2m^2)^2\beta,$$
$$c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_4^2}{\beta^3}},$$
$$A_4=\alpha^3-120[6m^2(1-m^2)+2(-1+2m^2)^2]\alpha\beta^2$$
$$+800(-1+2m^2)[18m^2(1-m^2)+4(-1+2m^2)^2]\beta^3.$$

Case 5

$$u=\frac{15c_1\beta[\alpha+20(2-m^2)\beta]}{A_5}-\frac{900c_1\beta^2}{A_5}\mathrm{cs}^2(\zeta),$$
$$\lambda=1-\frac{\alpha^2}{10\beta}+72(1-m^2)\beta-24(2-m^2)^2\beta,$$
$$c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_5^2}{\beta^3}},$$
$$A_5=\alpha^3-120[-6(1-m^2)+2(2-m^2)^2]\alpha\beta^2$$
$$+800(2-m^2)[-18(1-m^2)+4(2-m^2)^2]\beta^3.$$

Case 6

$$u=\frac{15c_1\beta[\alpha+20(2-m^2)\beta]}{A_6}-\frac{900c_1\beta^2}{A_6}\frac{\mathrm{sn}^2(\zeta)}{(1\pm\mathrm{dn}(\zeta))^2},$$

$$\lambda=1-\frac{\alpha^2}{10\beta}+\frac{9m^2\beta}{2}-6(-2+m^2)^2\beta,\quad c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_6^2}{\beta^3}},$$

$$A_6=\alpha^3-120\left[-\frac{3m^2}{8}+\frac{1}{2}(-2+m^2)^2\right]\alpha\beta^2+400(-2+m^2)\left[-\frac{9m^2}{8}+(-2+m^2)^2\right]\beta^3.$$

Case 7

$$u=\frac{15c_1\beta[\alpha+10(-2+m^2)\beta]}{A_7}+\frac{225c_1m^2\beta^2}{A_7}(\mathrm{sn}\zeta\pm\mathrm{i}\,\mathrm{cn}\zeta)^2$$

$$\lambda=1-\frac{\alpha^2}{10\beta}+\frac{9m^4\beta}{2}-6(-2+m^2)^2\beta,\quad c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_7^2}{\beta^3}},$$

$$A_7=\alpha^3-120\left[-\frac{3m^4}{8}+\frac{1}{2}(-2+m^2)^2\right]\alpha\beta^2+400(-2+m^2)\left[-\frac{9m^4}{8}+(-2+m^2)^2\right]\beta^3.$$

Case 8

$$u=\frac{15c_1\beta[\alpha+10(1-m^2)\beta]}{A_8}+\frac{225c_1\beta^2}{A_8}\frac{\mathrm{sn}^2\zeta}{(1\pm\mathrm{cn}\zeta)^2},$$

$$\lambda=1-\frac{\alpha^2}{10\beta}+\frac{9\beta}{2}-6(1-m^2)^2\beta,\quad c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_8^2}{\beta^3}},$$

$$A_8=\alpha^3-120\left[-\frac{3}{8}+\frac{1}{2}(1-m^2)^2\right]\alpha\beta^2+400(1-m^2)\left[-\frac{9}{8}+(1-m^2)^2\right]\beta^3.$$

Case 9

$$u=\frac{15c_1\beta[\alpha+10(1+m^2)\beta]}{A_9}+\frac{225c_1(-1+m^2)\beta^2}{A_9}\frac{\mathrm{dn}^2\zeta}{(1\pm m\,\mathrm{sn}\zeta)^2},$$

$$\lambda=1-\frac{\alpha^2}{10\beta}+\frac{9}{2}(-1+m^2)^2\beta-6(1+m^2)^2\beta,\quad c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_9^2}{\beta^3}},$$

$$A_9=\alpha^3-120\left[-\frac{3}{8}(-1+m^2)^2+\frac{1}{2}(1-m^2)^2\right]\alpha\beta^2+400(1+m^2)\left[-\frac{9}{8}(-1+m^2)^2+(1+m^2)^2\right]\beta^3.$$

Case 10

$$u=\frac{15c_1\beta[\alpha+10(1+m^2)\beta]}{A_{10}}+\frac{225c_1(1-m^2)\beta^2}{A_{10}}\frac{\mathrm{cn}^2\zeta}{(1\pm\mathrm{sn}\zeta)^2},$$

$$\lambda=1-\frac{\alpha^2}{10\beta}+\frac{9}{2}(1-m^2)(-1+m^2)\beta-6(1+m^2)^2\beta,\quad c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_{10}^2}{\beta^3}},$$

$$A_{10}=\alpha^3-120\left[-\frac{3}{8}(1-m^2)(-1+m^2)+\frac{1}{2}(1+m^2)^2\right]\alpha\beta^2+400(1+m^2)\left[-\frac{9}{8}(1-m^2)(-1+m^2)+(1+m^2)^2\right]\beta^3.$$

Case 11

$$u=\frac{15c_1\beta[\alpha+10(1+m^2)\beta]}{A_{11}}-\frac{225c_1\beta^2}{A_{11}}(m\,\mathrm{cn}\zeta\pm\mathrm{dn}\zeta)^2,$$

$$\lambda=1-\frac{\alpha^2}{10\beta}-\frac{9}{2}(-1+m^2)\beta-6(1+m^2)^2\beta,\quad c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_{11}^2}{\beta^3}},$$

$$A_{11}=\alpha^3-120\left[\frac{3}{8}(-1+m^2)+\frac{1}{2}(1+m^2)^2\right]\alpha\beta^2+400(1+m^2)\left[\frac{9}{8}(-1+m^2)+(1+m^2)^2\right]\beta^3.$$

Case 12

$$u=\frac{15c_1\beta[\alpha+10(1+m^2)\beta]}{A_{12}}+\frac{225c_1(1-m^2)^2\beta^2}{A_{12}}\frac{\mathrm{sn}^2\zeta}{(\mathrm{dn}\zeta\pm\mathrm{cn}\zeta)^2},$$
$$\lambda=1-\frac{\alpha^2}{10\beta}+\frac{9}{2}(1-m^2)^2\beta-6(1+m^2)^2\beta,\quad c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_{12}^2}{\beta^3}},$$
$$A_{12}=\alpha^3-120\left[-\frac{3}{8}(1-m^2)^2+\frac{1}{2}(1+m^2)^2\right]\alpha\beta^2+400(1+m^2)\left[-\frac{9}{8}(1-m^2)^2+(1+m^2)^2\right]\beta^3.$$

Case 13

$$u=\frac{15c_1\beta[\alpha+10(-2+m^2)\beta]}{A_{13}}+\frac{225c_1m^2\beta^2}{A_{13}}\frac{\mathrm{cn}^2\zeta}{\left(\sqrt{1-m^2}\pm\mathrm{dn}\zeta\right)^2},$$
$$\lambda=1-\frac{\alpha^2}{10\beta}+\frac{9m^2\beta}{2}\beta-6(-2+m^2)^2\beta,c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{A_{13}^2}{\beta^3}},$$
$$A_{13}=\alpha^3-120\left[-\frac{3m^2}{8}+\frac{1}{2}(-2+m^2)^2\right]\alpha\beta^2+400(-2+m^2)\left[-\frac{9m^2}{8}+(-2+m^2)^2\right]\beta^3.$$

Case 14

$$u=\frac{15c_1\beta}{\alpha^2}+\frac{900c_1\beta^2}{\alpha^3}\left(\frac{C}{\zeta}\right)^2,$$
$$\lambda=1-\frac{\alpha^2}{10\beta},\quad c_1=\frac{1}{15\sqrt{10}}\sqrt{-\frac{\alpha^6}{\beta^3}}.$$

In case of degeneracy, and due to the large number of solutions in Table 1, it is not advisable to treat every case. Therefore we shall only deal with a few cases as illustrative examples. If $m\to 1$, we can obtain the following soliton solutions from the Cases 1 and 2, respectively:

$$\begin{aligned}u&=\frac{15c_1\beta(\alpha-40\beta)}{\alpha^3-240\alpha\beta^2+3200\beta^3}\\&\quad+\frac{900c_1\beta^2}{\alpha^3-240\alpha\beta^2+3200\beta^3}\tanh^2(\zeta),\\\lambda&=1-\frac{\alpha^2}{10\beta}-24\beta,\\c_1&=\frac{1}{15\sqrt{10}}\sqrt{-\frac{(\alpha^3-240\alpha\beta^2+3200\beta^3)^2}{\beta^3}},\end{aligned}\tag{10}$$

$$\begin{aligned}u&=\frac{15c_1\beta(\alpha+20\beta)}{\alpha^3-240\alpha\beta^2+3200\beta^3}\\&\quad-\frac{900c_1\beta^2}{\alpha^3-240\alpha\beta^2+3200\beta^3}\mathrm{sech}^2(\zeta),\\\lambda&=1-\frac{\alpha^2}{10\beta}-24\beta,\\c_1&=\frac{1}{15\sqrt{10}}\sqrt{-\frac{(\alpha^3-240\alpha\beta^2+3200\beta^3)^2}{\beta^3}}.\end{aligned}\tag{11}$$

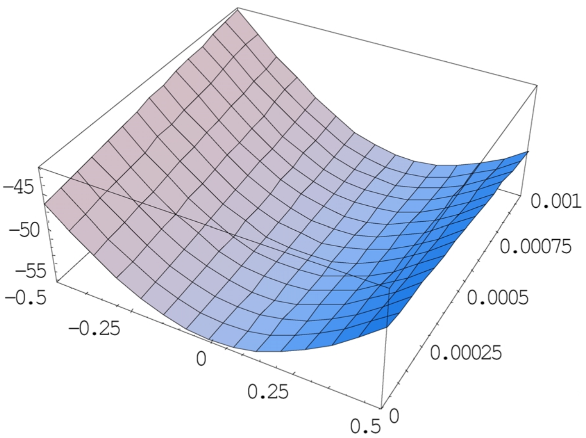

Fig. 1. A solitary wave solution of (11) is shown at $\alpha=10$, $\beta=-5$.

This solution is satisfactory when compared with that found previously by Sirendaoreji [19], see Figures 1 and 2.

If $m\to 0$, we can find the following trigonometric solutions from Cases 4 and 5, respectively:

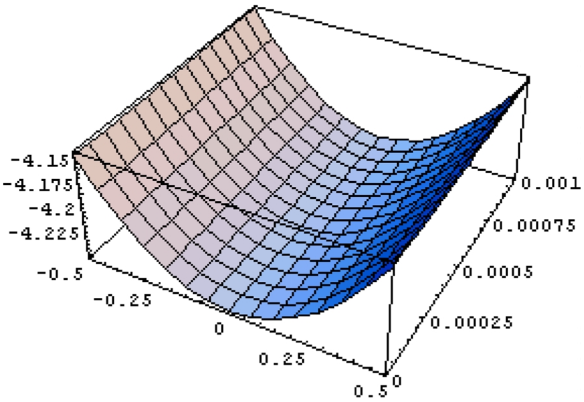

Fig. 2. A solitary wave solution by Sirendaoreji [19].

$$
\begin{aligned}
u &= \frac{15c_1\beta(\alpha-20\beta)}{\alpha^3-240\alpha\beta^2-3200\beta^3} \\
&\quad + \frac{900c_1\beta^2}{\alpha^3-240\alpha\beta^2-3200\beta^3}\csc^2(\zeta), \\
\lambda &= 1-\frac{\alpha^2}{10\beta}-24\beta, \\
c_1 &= \frac{1}{15\sqrt{10}}\sqrt{-\frac{(\alpha^3-240\alpha\beta^2-3200\beta^3)^2}{\beta^3}},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
u &= \frac{15c_1\beta(\alpha+40\beta)}{\alpha^3-240\alpha\beta^2-3200\beta^3} \\
&\quad + \frac{900c_1\beta^2}{\alpha^3-240\alpha\beta^2-3200\beta^3}\cot^2(\zeta), \\
\lambda &= 1-\frac{\alpha^2}{10\beta}-24\beta, \\
c_1 &= \frac{1}{15\sqrt{10}}\sqrt{-\frac{(\alpha^3-240\alpha\beta^2-3200\beta^3)^2}{\beta^3}}.
\end{aligned} \tag{13}
$$

I point out that all above solutions, except those from (10) to solution (13), are new and have not been reported in the literature. Moreover, only some solutions of (5) are shown.

4. Conclusion

Exact solutions of the modified Kawahara equation are studied by the mapping method. In this paper, we construct explicit and new solutions such as Jacobi elliptic solutions, combined Jacobi elliptic solutions, solitary wave solutions, periodic wave solutions and shock wave solutions, trigonometric solutions and rational solutions. The idea of our method is to use the elliptic equation involving four parameters instead of specific functions in previous methods [5 – 13]. Therefore, many tedious and repetitive calculations can be avoided. Moreover, this method can be applicable to a large variety of nonlinear partial differential equations.

Acknowledgements

I would like to express my sincere thanks to the referees for their useful comments.

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